FINITELY RELATED CLONES AND ALGEBRAS WITH CUBE-TERMS

PETAR MARKOVIĆ, MIKLÓS MARÓTI, AND RALPH MCKENZIE

Abstract. E. Aichinger, R. McKenzie, P. Mayr [1] have proved that every finite algebra with a cube-term (equivalently, with a parallelogram-term; equivalently, having few subpowers) is finitely related. Thus finite algebras with cube terms are inherently finitely related—every expansion of the algebra by adding more operations is finitely related. In this paper, we show that conversely, if $\mathbf{A}$ is a finite idempotent algebra and every idempotent expansion of $\mathbf{A}$ is finitely related, then $\mathbf{A}$ has a cube-term. We present further characterizations of the class of finite idempotent algebras having cube-terms, one of which yields, for idempotent algebras with finitely many basic operations and a fixed finite universe A, a polynomial-time algorithm for determining if the algebra has a cube-term. We also determine the maximal non-finitely related idempotent clones over A. The number of these clones is finite.

1. Introduction

An algebra $\mathbf{A}$ is said to be *finitely related* if there are finitely many finitary relations $\rho_1, \ldots, \rho_n$ over A (equivalently, there is a finitely relation ρ over A) such that any operation F on A is a term operation of $\mathbf{A}$ if and only if F respects each of the relations $\rho_1, \ldots, \rho_n$ (equivalently, F respects ρ). A *cube operation* over a set A is an operation $c(x_0, \ldots, x_{n-1})$ such that for each $0 \leq i < n$, the algebra $\langle A, c \rangle$ satisfies an equation $c(w_0, \ldots, w_{n-1}) = x$ where $\{w_0, \ldots, w_{n-1}\} \subseteq \{x, y\}$ and $w_i = y$. (Here x and y are distinct variables.) An algebra $\mathbf{A}$ is said to have a cube-term if its clone of term operations contains a cube operation. An *idempotent operation* over A is a function $f : A^n \to A$ for some n such that the function $g(x) = f(x, \ldots, x)$ is the identity function on A. By an *idempotent algebra* we mean an algebra whose basic operations are idempotent (and thus every term operation of the algebra is idempotent).

In E. Aichinger, R. McKenzie, P. Mayr [1] it was proved that every finite algebra with a cube-term is finitely related. It is known (see [5]) that every algebra with a cube-term generates a congruence modular variety; equivalently, such an algebra possesses a sequence of Day operations. M. Valeriote has conjectured that the result of Aichinger, McKenzie and Mayr has a converse: every finite algebra in a congruence-modular variety, if it is finitely related, must have a cube-term. A special case of this, which had earlier been conjectured by L. Zádori, has been established just this year by L. Barto [2] (see also P. Marković and R. McKenzie [10]): A finite algebra in a congruence-distributive variety is finitely related iff it has a near-unanimity operation. We hope that ultimately, some of the characterizations

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of algebras with cube-terms that are mentioned or proved in this paper will be of use in settling M. Valeriote's conjecture.

2. Idempotent algebras with a cube-term

Let $\mathbf{A}$ be a finite algebra. A *cube-term blocker* in $\mathbf{A}$ is any pair (D, S) of subuniverses of $\mathbf{A}$ such that $\emptyset < D < S \leq \mathbf{A}$ and such that for every term operation $t(x_1, \ldots, x_n)$ of $\mathbf{A}$ there is i, $1 \leq i \leq n$, so that whenever $\bar{s} \in S^n$ and $s_i \in D$ then $t(\bar{s}) \in D$.

Theorem 2.1. *Let* $\mathbf{A}$ *be a finite idempotent algebra. Then* $\mathbf{A}$ *has a cube-term iff it possesses no cube-term blockers.*

In order to establish this theorem, we need some auxiliary definitions and results.

Definition 2.2. Given elements a, b, c in an idempotent algebra $\mathbf{B}$, we shall write $c \prec (a, b)$ to denote that there is a term $t(x_1, \ldots, x_n)$ for some $n \geq 1$ and a set of equations true in $\mathbf{B}$ of the form $t(\bar{u}) = c$ with $\bar{u} \in \{a, b\}^n$, namely

$$\begin{array}{rcl} t(\bar{u}^1) & = & c \\ t(\bar{u}^2) & = & c \\ & \vdots & \\ t(\bar{u}^m) & = & c\,, \end{array}$$

such that for each $1 \leq i \leq n$ there is $1 \leq j \leq m$ so that $\bar{u}^j_i = b$. We write $a \prec b$ to denote that $a \prec (a, b)$. When $a \prec (a, b)$ via t, we say that t is a *cube-term for* $\{(a, b)\}$. We say that t is a cube-term for $\{(a_1, b_1), \ldots, (a_k, b_k)\} \subseteq B^2$ if t is a cube-term for $\{(\bar{a}, \bar{b})\}$ in $\mathbf{B}^k$ where $\bar{a} = (a_1, \ldots, a_k)$ and $\bar{b} = (b_1, \ldots, b_k)$.

We define $c \in \langle a, b\rangle$ via t to mean that there is a tuple $\bar{u}$ with range contained in $\{a, b\}$ so that $t(\bar{u}) = c$.

By a *cube-term* for an algebra $\mathbf{A}$ we will mean a term $t(x_1, \ldots, x_k)$ such that where $\mathbf{F}' = \mathbf{F}'(x, y)$ is the idempotent reduct of the free algebra over the variety generated by $\mathbf{A}$ with free generators x, y, we have that $x \prec y$ in $\mathbf{F}'$ via t. Note that what we have defined as a cube-term is the same as what is called in [5] a "special cube-term" for $\mathbf{A}$.

If $p = p(x_1, \ldots, x_e)$ and $q = q(x_1, \ldots, x_f)$ are terms, then by $p \star q$ we denote the term

$$p(q(x_{11}, \ldots, x_{1f}), q(x_{21}, \ldots, x_{2f}), \ldots, q(x_{e1}, \ldots, x_{ef}))\,,$$

with ef many distinct variables. The following assertions are easy to prove.

Lemma 2.3. *In any idempotent algebra* $\mathbf{B}$ *the following are true.*

1. *Suppose that t is an n-ary term, $t_1, \ldots, t_n$ are terms, and for $1 \leq i \leq n$, $c_i \prec (a, b)$ via t_i and $t(c_1, \ldots, c_n) = c$. Then $c \prec (a, b)$ via $t(t_1, \ldots, t_n)$.*
2. *Suppose that $t_1, \ldots, t_k$ are terms, $t = t_1 \star t_2 \star \cdots \star t_k$, and $a, b, c \in A$. If for some i, $c \prec (a, b)$ via t_i, then $c \prec (a, b)$ via t. If for some i, $c \in \langle a, b\rangle$ via t_i then $c \in \langle a, b\rangle$ via t.*
3. *If $\mathbf{B}$ is finite, then there exists a term $m(x_1, \ldots, x_p)$ such that whenever $a, b, c \in A$ and $c \prec (a, b)$ (respectively $c \in \langle a, b\rangle$), then $c \prec (a, b)$ via m (respectively, $c \in \langle a, b\rangle$ via m).*

We make a slight digression to harvest some interesting facts.

Lemma 2.4. *Let* $\mathbf{B}$ *be a finite idempotent algebra with a congruence* θ*. If* $\mathbf{B}/\theta$ *has a cube-term* t_1*, and there is a term* t_2 *which is a cube-term for the algebra* $\mathbf{S}$ *that* $\mathbf{B}$ *induces on* S *for every* θ*-equivalence class* S*, then* $s = t_2 \star t_1$ *is a cube-term for* $\mathbf{B}$*.*

Proof. Say $t_2 = t_2(x_1, \ldots, x_k)$ and $t_1 = t_1(x_1, \ldots, x_\ell)$, and where $M = \{ij : 1 \leq i \leq k, 1 \leq j \leq \ell\}$ $s = t_2 \star t_1 = s(x_{ij} : ij \in M)$.

To see that s is a cube-term for $\mathbf{B}$, note that we have $\{\bar{z}^1, \ldots, \bar{z}^m\} \subseteq \{x, y\}^k$ so that the equations $t_2(\bar{z}^i) = x$ are laws in every θ-equivalence class; and we have $\{\bar{w}^1, \ldots, \bar{w}^n\} \subseteq \{x, y\}^\ell$ so that the equations $t_1(\bar{w}^i) = x$ are laws in $\mathbf{B}/\theta$; and such that for every $1 \leq i \leq k$ there is $1 \leq j \leq m$ with $z_i^j = y$, and for every $1 \leq i \leq \ell$ there is $1 \leq j \leq n$ with $z_i^j = y$. For each $1 \leq u \leq m$, $1 \leq v \leq n$ we define $\bar{\tau}^{uv} \in \{x, y\}^M$. Namely, we put $\bar{\tau}_{ij}^{uv} = y$ iff $z_i^u = y = w_j^v$, and we put $\bar{\tau}_{ij}^{uv} = x$ elsewhere.

Now let $a, b \in B$ and $1 \leq u \leq m$, $1 \leq v \leq n$. Write $p(x, y) = s(\bar{\tau}^{uv})$. We want to prove that $p(a, b) = a$. Write $q^v(x, y) = t_1(\bar{z}^v)$ and $q^u(x, y) = t_2(\bar{z}^u)$. It is easy to see that $p(a, b) = q^u(a, q^v(a, b))$. Now $q^v(a, b)$ and a belong to one θ-equivalence class T, since $\mathbf{B}/\theta \models p(x, y) = x$. It follows that $p(a, b) = a$ since $\mathbf{T} \models q^u(x, y) = x$.

Finally, it is easy to see that for every $1 \leq i \leq k$ and $1 \leq j \leq \ell$, there is $1 \leq u \leq m$ and $1 \leq v \leq n$ such that $\tau_{ij}^{uv} = y$. Thus indeed, s is a cube-term for $\mathbf{B}$. •

Corollary 2.5. *Let* $\mathbf{B}_1, \ldots, \mathbf{B}_n$ *be similar finite idempotent algebras. If each of these algebras has a cube-term then there is a term* t *which is a cube-term for all of the algebras.*

Proof. Assume that each $\mathbf{B}_i$ has a cube-term. It suffices to prove that for two finite idempotent algebras $\mathbf{B}, \mathbf{C}$, if both have cube-terms, then $\mathbf{B} \times \mathbf{C}$ has a cube-term. For then, inductively, we find that there must be a cube-term for $\mathbf{B}_1 \times \cdots \times \mathbf{B}_n$.

So let $\mathbf{B}$ have cube-term t_1 and $\mathbf{C}$ have cube-term t_2. Then the algebra $\mathbf{B} \times \mathbf{C}$ has a congruence θ, namely the first projection congruence, so that t_1 is a cube-term for $(\mathbf{B} \times \mathbf{C})/\theta$, and t_2 is a cube-term for every θ-block. (The block algebras are all isomorphic to $\mathbf{C}$.) Thus this corollary follows from Lemma 2.4. •

The next corollary obviously follows from Lemma 2.4 and Corollary 2.5.

Corollary 2.6. *Let* $\mathbf{B}$ *be a finite idempotent algebra with a congruence* θ*. If* $\mathbf{B}/\theta$ *has a cube-term and every* θ*-equivalence class has a cube-term, then* $\mathbf{B}$ *has a cube-term.*

Corollary 2.7. *Let* $\mathbf{B}$ *be a finite idempotent algebra.* $\mathbf{B}$ *has a cube-term iff every simple algebra* $\mathbf{S} \in HS(\mathbf{B})$ *has a cube-term.*

Proof. If $\mathbf{B}$ has a cube-term then of course, every algebra in $HS(\mathbf{B})$ has a cube-term. If $\mathbf{B}$ has no cube-term, then let $\mathbf{S}$ be an algebra of least cardinality in $HS(\mathbf{B})$ that has no cube-term. Then $|S| > 1$ and it follows from Corollary 2.6 that $\mathbf{S}$ is simple. •

Now we return to the task of proving that an idempotent algebra has a cube-term iff it has no cube-term blocker.

Theorem 2.8. *Let* $\mathbf{B}$ *be a finite idempotent algebra. Then* $\mathbf{B}$ *has a cube-term if and only if for all* $a, b \in B$ *we have* $a \prec b$ *in* $\mathbf{B}$*.*

Proof. We assume that for all $a, b \in B$ we have $a \prec b$. The term m from Lemma 2.3(3) thus has the property that $a \prec b$ via m for all $a, b \in B$. But m is most likely not yet a cube-term for $\mathbf{B}$.

We prove by induction on $|E|$ that for every nonvoid set $E \subseteq B^2$, there is a cube-term for E (recall that this notion was defined in Definition 2.2). The base step, $|E| = 1$, is our initial hypothesis. For the induction step, suppose that every set $E \subseteq B^2$ with $|E| \le r$ has a cube-term and let $S \subseteq B^2$ with $|S| = r+1$, $S = E \cup \{(c,d)\}$ where $|E| = r$. Choose a term $t(x_1, \ldots, x_n)$ and tuples $\bar{z}^1, \ldots, \bar{z}^k \in \{x,y\}^n$ to witness that t is a cube-term for E.

For $1 \le i \le k$ choose a term s_i so that where $t_i(x,y) = t(\bar{z}^i)$, we have $c \prec t_i(c,d)$ via s_i. Say $s_i = s_i(x_1, \ldots, x_{n_i})$ and let $P_i = \{\bar{z}^{i1}, \ldots, \bar{z}^{ik_i}\} \subseteq \{x,y\}^{n_i}$ be such that where $\bar{e}^j \in B^{n_i}$ results from $\bar{z}^{ij}$ by mapping $x \mapsto c$ and $y \mapsto t_i^{\mathbf{B}}(c,d)$, we have equations valid in $\mathbf{B}$—$s_i(\bar{e}^1) = c, \ldots, s_i(\bar{e}^{k_i}) = c$—witnessing that $c \prec t_i(c,d)$ via s_i.

We claim that the term $t' = s_1 \star s_2 \star \cdots \star s_k \star t$ is a cube-term for $S = E \cup \{(c,d)\}$. The term t' has $m = n_1 n_2 \cdots n_k n$ variables, namely x_α where α ranges over the set M of all words $i_1 i_2 \cdots i_k i$ with $1 \le i_j \le n_j$ for $1 \le j \le k$, and with $1 \le i \le n$. To see that t' is a cube-term for S, we have to define an appropriate set $P \subseteq \{x,y\}^M$.

We take P to be the set of all functions in $\{x,y\}^M$ constructed as follows. Choose an arbitrary u with $1 \le u \le k$, and then choose any v with $1 \le v \le k_u$. We define a function $\bar{w}^{uv} \in \{x,y\}^M$ like this. For any $\alpha = i_1 \cdots i_k j \in M$, put $\bar{w}^{uv}_\alpha = y$ if $\bar{z}^{uv}_{i_u} = y$ and $\bar{z}^u_j = y$, and put $\bar{w}^{uv}_\alpha = x$ otherwise. Then we take

$$P = \{\bar{w}^{uv} : 1 \le u \le k, 1 \le v \le k_u\}.$$

It is easy to see that for every $\alpha \in M$, there is $\bar{w}^{uv} \in P$ with $\bar{w}^{uv}_\alpha = y$. It is a straightforward, if tedious, calculation to show that for every $\bar{w}^{uv} \in P$, where $t_{uv}(x,y) = t'(\bar{w}^{uv})$, we have that $\mathbf{B} \models t_{uv}(c,d) = c$; and also to show that $\mathbf{B} \models t_{uv}(a,b) = a$ for every $(a,b) \in E$. This means that t' is a cube-term for S, as desired.

Thus the inductive proof is complete. It follows that there is a cube-term t for B^2. This is the same thing as a cube-term for $\mathbf{B}$. •

Proof of Theorem 2.1. It should be clear that an algebra with a cube-term cannot have a cube-term blocker. Let $\mathbf{B}$ be a finite idempotent algebra with no cube-term. Let $\mathbf{A}$ be minimal under inclusion among all the subalgebras of $\mathbf{B}$ that have no cube-term. It will suffice to show that $\mathbf{A}$ has a cube-term blocker (which will also be a cube-term blocker in $\mathbf{B}$).

We have that $\mathbf{A}$ has no cube-term, but every proper subalgebra of $\mathbf{A}$ does have a cube-term. Thus by Theorem 2.8, there is at least one pair $(a,b) \in A^2$ such that $a \not\prec b$; moreover, for every such pair we have that $\{a,b\}$ generates $\mathbf{A}$. It is well-known, and follows easily from M. Valeriote [11], Proposition 3.1, that if tame-congruence type **1** occurs in the variety generated by $\mathbf{A}$, then since $\mathbf{A}$ is idempotent, there must be a subalgebra $\mathbf{S} \le \mathbf{A}$ and congruence θ of $\mathbf{S}$ such that $\mathbf{S}/\theta$ is a two-element algebra in which every operation is a projection. In this case, choosing D to be either one of the θ-equivalence classes, (D,S) is a cube-term blocker in $\mathbf{A}$, and we are done. So we can assume that type **1** does not occur. Then by L. Barto, M. Kozik [3], the algebra $\mathbf{A}$ has a cyclic term. This is a quite deep result that we will need to use in the middle of our proof.

Claim 1: For $a, b, c \in A$, we have that $a \prec b \prec c$ implies $a \prec c$.

To see this, note that we can assume that $\{a, c\}$ generates $\mathbf{A}$, else $a \prec c$ is guaranteed. Let $b = s(a, c)$ where $s(x, y)$ is a term. Suppose that $t_1(x_1, \ldots, x_n)$ is a cube-term for (a, b) witnessed by equations $t_1(\bar{c}^j) = a$ $(1 \leq i \leq k)$, and $t_2(x_1, \ldots, x_m)$ is a cube-term for (b, c) witnessed by equations $t_2(\bar{d}^j) = b$ $(1 \leq j \leq \ell)$. Let $t = t_1 \star t_2 \star s$. Thus t has variables x_{ijk} with i ranging over $\{1, \ldots, n\}$, j ranging over $\{1, \ldots, m\}$ and k ranging over $\{1, 2\}$.

It is easy to check that t is a cube-term for (a, c).

Claim 2: For $b \in A$, the set $\{x : x \prec b\}$ is a subalgebra of $\mathbf{A}$.

To see this, let $s(c_1, \ldots, c_k) = e$ where $c_i \prec b$ for all i and s is a term. To show that $e \prec b$, we can assume that $\{b, e\}$ generates $\mathbf{A}$. Thus for $1 \leq i \leq k$ we can choose a term $s_i(x, y)$ so that $s_i(b, e) = c_i$. Also, for $1 \leq i \leq k$, a term $t_i(x_1, \ldots, x_{n_i})$ which is a cube-term for (c_i, b). Now put

$$t = s(t_1(s_1(x_{111}, x_{112}), \ldots, s_1(x_{1n_11}, x_{1n_12})), \ldots,$$
$$\ldots, (t_k(s_k(x_{k11}, x_{k12}), \ldots, s_1(x_{kn_k1}, x_{kn_k2}))) \, .$$

It can be easily checked that t is a cube-term for $\{(e, b)\}$.

Claim 3: Suppose that $a \prec b$ and c is in the subalgebra generated by $\{a, b\}$, then $c \prec b$.

This follows from Claim 2, since $b \prec b$.

Claim 4: For each $x \in A$, either $x \prec y$ for all $y \in A$, or else $y \prec x$ for all $y \in A$. We put D equal to the set of all $x \in A$ for which the first alternative holds, and U equal to the set of all x satisfying the second alternative. Thus $A = D \cup U$.

To prove it, let $c(x_1, \ldots, x_n)$ be a cyclic term for $\mathbf{A}$. This simply means that $n > 1$ and this term satisfies in $\mathbf{A}$ the equation $c(x_1, \ldots, x_n) = c(x_2, x_3, \ldots, x_n, x_1)$ (and of course the equation $c(x, x, \ldots, x) = x$). As we remarked at the beginning of this proof, the existence of such a term for $\mathbf{A}$ follows from the main result of L. Barto, M. Kozik [3]. Let $a \in A$. Suppose that for some $b \in A$, it is not the case that $a \prec b$. Then put $e = c(b, a, \ldots, a)$. It is not the case that $a \prec e$, for if t is a cube-term for (a, e), then $t \star c$ is a cube-term for (a, b), due to the cyclic equation that c satisfies. Thus $\{a, e\}$ generates $\mathbf{A}$. Then since we can write $b = s(e, a)$ for some term s, it follows that $c \star s$ is a cube-term for $\{(e, a)\}$. Thus $e \prec a$ and, as we saw,$\{a, e\}$ generates $\mathbf{A}$, implying by Claim 3 that $x \prec a$ for all x.

Claim 5: $D \cap U = \emptyset$.

To see this, suppose that $p \in D \cap U$. For any $a, b \in A$ we have $a \prec p \prec b$, giving $a \prec b$ by Claim 1. Then by Lemma 2.8, $\mathbf{A}$ has a cube-term. But this is a contradiction.

Whenever $a \prec b$ fails, then $a \in U$ and $b \in D$. Thus both D and U are non-void.

Claim 6: For $a, b \in A$ we have $a \prec b$ iff $a \in D$ or $b \in U$.

To see it, suppose that $a \in U$ and $b \in D$ and $a \prec b$. Then by Claim 1 and the definition of D and U, we have that $\{a, b\} \subseteq D \cap U$. This contradicts the previous claim.

Claim 7: Let $a, d \in D$ and $b \in U$. Then $d \prec (a, b)$ and $d \prec (b, a)$.

To see this, let $e = c(a, b, b, \ldots, b)$ where $c(x_1, \ldots, x_n)$ is a cyclic term for $\mathbf{A}$. Thus $e \prec (a, b)$ and $e \prec (b, a)$ via $c(\bar{x})$. It follows that $e \prec (a, b)$ and $e \prec (b, a)$ via the term m of Lemma 2.3(3). Also $d \prec e$ via m since $d \in D$, and $d \in \langle a, b\rangle$ via m since $\{a, b\}$ generates $\mathbf{A}$. Putting these facts together, one can easily verify that $d \prec (a, b)$ and $d \prec (b, a)$ via $m \star m \star c$.

We conclude this proof of Theorem 2.1 by showing that (D, A) is a cube-term blocker for $\mathbf{A}$. Note that it follows from Claim 2 that D is a subuniverse of $\mathbf{A}$.

Claim 8: For every term $t = t(x_1, \ldots, x_n)$, there is an i, $1 \le i \le n$, so that $t(A^{i-1} \times D \times A^{n-i}) \subseteq D$.

Suppose that this fails for a term $t = t(x_1, \ldots, x_n)$. Choose $a \in D$ and $b \in U$. We shall show that $b \prec a$, contradicting Claim 6. For each $1 \le i \le n$, choose $\bar{w}^i \in A^n$ such that $w^i_i \in D$ and $t(\bar{w}^i) = u_i \in U$. Let m be a term for $\mathbf{A}$ with the properties specified in Lemma 2.3(3). Then we have that $w^i_j \in \langle a, b\rangle$ via m and $b \prec u_i$ via m, for all $1 \le i, j \le n$. Moreover, $w^i_i \prec (b, a)$ via m by Claim 7 and Lemma 2.3(3).

Consequently, where $r = m \star t \star m$ the reader can verify that $b \prec a$ via r. This contradicts Claim 6 and the contradiction proves Claim 8.

We have now shown that (D, A) is a cube-term blocker for $\mathbf{A}$, thus completing our proof of Theorem 2.1.

3. Finitely related idempotent algebras

An algebra $\mathbf{A}$ is said to be an *expansion* (or *polynomial expansion*) of an algebra $\mathbf{B}$ if these algebras have the same universe and every operation of $\mathbf{B}$ is a term operation of $\mathbf{A}$ (or every operation of $\mathbf{B}$ is a term operation of $\mathbf{A}$ and every operation of $\mathbf{A}$ is a polynomial operation of $\mathbf{B}$). The class of finite algebras having a cube-term is a family of finitely related finite algebras closed under expansions. We shall see that the class of finite idempotent algebras having a cube-term is just the largest class of finitely related, finite, idempotent algebras closed under idempotent expansions.

Let A be a finite set. A *clone of operations* over A is a set of finitary operations $f : A^n \to A$ (for some non-negative integer n) that is closed under compositions and contains all the trivial (projection) operations $p^n_i : (x_1, \ldots, x_n) \mapsto x_i$. The set of all idempotent operations over A is a clone, denoted $\mathcal{I}$. The set of all clones over A is a lattice under set-inclusion. This lattice is both algebraic and dually algebraic, and is dually isomorphic to the lattice of all relational clones over A. The clone $\mathcal{I}$ is a compact (finitely generated) element of the lattice of clones, and is also co-compact (finitely related), since it is identical with the set of operations that respect each of the unary relations $r_a = \{(a)\}$, $a \in A$. We work in the lattice of idempotent clones over A. This is a lower interval in the lattice of all clones, and is itself an algebraic and dually algebraic lattice. We note that the idempotent clones over A are precisely the clones of term operations of the idempotent algebras with universe A. We now define what prove to be precisely the maximal non-finitely related (not co-compact) elements of the lattice of idempotent clones.

Definition 3.1. Let A be a finite set and suppose that $\emptyset \ne D \subset S \subseteq A$ (i.e., S and D are distinct non-void subsets of A with D included in S). For $n \ge 1$ we define

$R_n(D,S)$ to be the set of all $\bar{x} = (x_1, \ldots, x_n) \in S^n$ such that for at least one i, $1 \leq i \leq n$, we have $x_i \in D$. In other words,

$$R_n(D,S) = S^n \setminus (S \setminus D)^n \,.$$

We define $\mathcal{C}_{\mathcal{I}}(D,S)$ (or $\mathcal{C}(D,S)$) to be the clone of all idempotent operations (or of all operations) on A that respect all the relations $R_n(D,S)$, $n \geq 1$.

Lemma 3.2. *Let* $\mathbf{A}$ *be a finite algebra with universe* A *and suppose that* $\emptyset \neq D \subset S \subseteq A$. *Then* (D,S) *is a cube-term blocker for* $\mathbf{A}$ *iff all operations of* $\mathbf{A}$ *belong to the clone* $\mathcal{C}(D,S)$.

Proof. The proof that if (D,S) is a cube-term blocker for $\mathbf{A}$ then all operations of $\mathbf{A}$ belong to the clone $\mathcal{C}(D,S)$ is easy, and left to the reader. Suppose, on the other hand, that all basic operations of $\mathbf{A}$, and thus all term operations of $\mathbf{A}$, respect all of the relations $R_n(D,S)$, $n \geq 1$. Since the projection at the first variable of $R_1(D,S)$ is the set D, and of $R_2(D,S)$ is the set S, then both D and S are subuniverses of $\mathbf{A}$. To see that (D,S) is a cube-term blocker for $\mathbf{A}$, let $t = t(x_1, \ldots, x_m)$ be a term operation of $\mathbf{A}$. To get a contradiction, suppose that for each $1 \leq i \leq m$ there is an m-tuple $\bar{w}^i \in S^m$ with $w^i_i \in D$ such that $t(\bar{w}^i) \notin D$. Now the m-tuples $\bar{z}^j$, $1 \leq j \leq m$, with $z^j_i = w^i_j$, all belong to $R_m(D,S)$ (since $z^j_j \in D$) and, operating in the algebra $\mathbf{A}^m$, we have that

$$t^{\mathbf{A}^m}(\bar{z}^1, \ldots, \bar{z}^m) \notin R_m(D,S)\,;$$

i.e., the operation t does not respect the relation $R_m(D,S)$. This is a contradiction. •

Lemma 3.3. *Let* A *be a finite set and* $\emptyset \neq D \subset S \subseteq A$. *Each of the clones* $\mathcal{C}_{\mathcal{I}}(D,S)$ *and* $\mathcal{C}(D,S)$ *fails to be finitely related. Moreover, if also* $\emptyset \neq D' \subset S' \subseteq A$, *then either of* $\mathcal{C}(D,S) \subseteq \mathcal{C}(D',S')$ *or* $\mathcal{C}_{\mathcal{I}}(D,S) \subseteq \mathcal{C}_{\mathcal{I}}(D',S')$ *implies* $D = D'$ *and* $S = S'$.

Proof. If $\mathcal{C}(D,S)$ were finitely related, then $\mathcal{C}_{\mathcal{I}}(D,S)$ would be finitely related as well, since $\mathcal{I}$ is finitely related. If $\mathcal{C}_{\mathcal{I}}(D,S)$ is finitely related, then it is co-compact, so that finitely many of the relations $R_n(D,S)$, together with the relations r_a ($a \in A$) define $\mathcal{C}_{\mathcal{I}}(D,S)$. Thus, suppose that $\mathcal{C}_{\mathcal{I}}(D,S)$ is identical with the set of idempotent operations over A that respect $R_1(D,S), \ldots, R_k(D,S)$. We get a contradiction by constructing an idempotent operation $f = f(x_1, \ldots, x_{k+1})$ that respects $R_j(D,S)$ for $j \leq k$, but does not respect $R_{k+1}(D,S)$. Namely, define $f(\bar{x})$ to be x_{i_0} where i_0 is the least i with $x_i \notin D$ if for at most one i, $1 \leq i \leq k+1$, is $x_i \in D$; while take $f(\bar{x})$ to be x_{i_0} where i_0 is the least i with $x_i \in D$ if there exist $1 \leq i < j \leq k+1$ with $\{x_i, x_j\} \subseteq D$.

This operation is clearly idempotent. Choosing $a \in D$ and $b \in S \setminus D$ and considering the tuples $\bar{v}^i = (b, \ldots, b, a, b, \ldots, b)$ where the i-th entry is a, we find that $f(\bar{v}^1, \ldots, \bar{v}^{k+1}) = \bar{b}$, showing that f does not respect $R_{k+1}(D,S)$. On the other hand, if $1 \leq m \leq k$ and $\bar{w}^1, \ldots, \bar{w}^{k+1} \in R_m(D,S)$, then for each j between 1 and $k+1$ choose i_j between 1 and m with $w^j_{i_j} \in D$. Now we can find $1 \leq j_1 < j_2 \leq k+1$ with $i_{j_1} = i_{j_2} = i$. Then the i-th entry of $f(\bar{w}^1, \ldots, \bar{w}^{k+1}) = \bar{u}$ belongs to D and $\bar{u} \in R_m(D,S)$. Thus f respects $R_m(D,S)$.

Thus all of these clones are non-finitely related. The proof that $\mathcal{C}_{\mathcal{I}}(D,S) \subseteq \mathcal{C}_{\mathcal{I}}(D',S')$ implies $D = D'$ and $S = S'$ proceeds by considering cases. If $D \not\subseteq D'$ then choosing $d \in D \setminus D'$ and putting $f(x,y) = d$ unless $x = y$ in which case

$f(x,y) = x$, gives an idempotent operation f which clearly belongs to $\mathcal{C}_{\mathcal{I}}(D,S) \setminus \mathcal{C}_{\mathcal{I}}(D',S')$. Thus $D \subseteq D'$.

If $D' \not\subseteq S$ then choosing $d' \in D' \setminus S$ and $b \in A \setminus D'$ and putting $f(x,y) = b$ whenever $x \neq y$ and $d' \in \{x,y\}$, and $f(x,y) = x$ for all other pairs x,y, gives an operation f that contradicts $\mathcal{C}_{\mathcal{I}}(D,S) \subseteq \mathcal{C}_{\mathcal{I}}(D',S')$. Thus we have $D \subseteq D' \subseteq S$.

Now if $D' \neq D$, choose $d' \in D' \setminus D$ and $b \in A \setminus S'$ and define $f(x,y) = b$ if $d' \in \{x,y\} \not\subseteq S$, $f(x,y) = z$ where $\{x,y\} = \{z,d'\}$ if $x \neq y$ and $d' \in \{x,y\} \subseteq S$, and put $f(x,y) = x$ for all other pairs x,y. This again works.

Thus we have that $D = D'$. Now suppose that $S \not\subseteq S'$ and choose $b \in S \setminus S'$. Of course, $b \notin D$. Define $f(x,x) = x$ and $f(x,y) = x$ if $x \in D$ and $y \in S$ and for all other pairs x,y put $f(x,y) = b$. Then where $d \in D$ and $c \in S' \setminus D$, we have that $f(c,d) = b \notin S'$, so $f \notin \mathcal{C}_{\mathcal{I}}(D',S')$. It is easy to see that $f \in \mathcal{C}_{\mathcal{I}}(D,S)$. Thus $S \subseteq S'$. Finally, suppose that $b \in S' \setminus S$. Define $f(x,x) = x$ and $f(x,y) = x$ if $\{x,y\} \subseteq S$, and set $f(x,y) = b$ for all other pairs x,y. This operation again belongs to $\mathcal{C}_{\mathcal{I}}(D,S) \setminus \mathcal{C}_{\mathcal{I}}(D',S')$. •

Theorem 3.4. *Let* $\mathbf{A}$ *be a finite idempotent algebra. The following are equivalent:*

1. $\mathbf{A}$ *has a cube-term.*
2. *For every* $\emptyset \neq D \subset S \subseteq A$*, there is some basic operation of* $\mathbf{A}$ *that fails to belong to* $\mathcal{C}_{\mathcal{I}}(D,S)$*.*
3. *Every idempotent expansion of* $\mathbf{A}$ *is finitely related.*
4. *Every expansion of* $\mathbf{A}$ *is finitely related*

Proof. That (1) $\Rightarrow$ (4) is a consequence of the the main result of Aichinger, Mayr and McKenzie [1]. (4) $\Rightarrow$ (3) trivially, and (3) $\Rightarrow$ (2) by Lemma 3.3. That (2) $\Rightarrow$ (1) follows from Theorem 2.1 and Lemma 3.2. •

Corollary 3.5. *Let* A *be a finite non-void set. Every non-finitely related idempotent clone on* A *is contained in one of the clones* $\mathcal{C}_{\mathcal{I}}(D,S)$*. These clones are the maximal members of the set of non-finitely related idempotent clones.*

Proof. Let $\mathcal{C}$ be a non-finitely related idempotent clone on A. By Theorem 3.4, applied to the algebra $\langle A, f (f \in \mathcal{C})\rangle$, we have that $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{I}}(D,S)$ for some pair $\emptyset \neq D \subset S \subseteq A$. From Lemma 3.3, $\mathcal{C}_{\mathcal{I}}(D,S)$ is non-finitely related. If $\mathcal{C}_{\mathcal{I}}(D,S) \subset \mathcal{D} \subseteq \mathcal{I}$, then again by Lemma 3.3, the clone $\mathcal{D}$ can be contained in no one of the clones $\mathcal{C}_{\mathcal{I}}(D',S')$, and then it follows from Theorem 3.4 that $\mathcal{D}$ is finitely related. Thus $\mathcal{C}_{\mathcal{I}}(D,S)$ is maximal among the non-finitely related idempotent clones. •

Corollary 3.6. *Let* A *be a finite non-void set. There is a polynomial-time algorithm to input any finite sequence* $f_1, \ldots, f_n$ *of idempotent operations over* A *and output the correct answer to the question: does the algebra* $\langle A, f_1, \ldots, f_n\rangle$ *have a cube-term.*

Proof. Let $(D_1,S_1), \ldots, (D_M,S_M)$ be a list, without repetitions, of all the pairs (D,S) of subsets of A with $\emptyset \neq D \subset S$. Let $\mathcal{C}^j = \mathcal{C}_{\mathcal{I}}(D_j,S_j)$ for $1 \leq j \leq M$. By Theorem 3.4, an idempotent algebra $\langle A, f_1, \ldots, f_n\rangle$ has a cube-term iff for no j is $\{f_1, \ldots, f_n\} \subseteq \mathcal{C}^j$. We can test whether $f_i \in \mathcal{C}^j$ in polynomial time. The condition $f_i \in \mathcal{C}^j$ holds iff f_i has an input variable x_u such that whenever $\bar{w}$ is a tuple in the domain of f_i and all entries of $\bar{w}$ belong to S_j and $w_u \in D_j$, then $f_i(\bar{w}) \in D_j$. With one pass through the operation table of f_i, we can compile a list of the variables x_u for which this is true, and then simply check if this list is non-empty.

Another simple algorithm, using the above as sub-routine, compiles a list of all the pairs (i, j) such that $f_i \in \mathcal{C}^j$. Then we simply have to check if there is $j = j_0$ such that (i, j_0) is on this list for all $1 \leq i \leq n$. •

Discussion. Jonah Horowitz [8] has shown that it is decidable to determine if a finite algebra of finite signature has a cube-term; but even for idempotent algebras, it is not known whether there is a polynomial-time algorithm to determine if $\langle A, f_1, \ldots, f_n \rangle$ has a cube-term. Our algorithm for fixed A and variable list of idempotent operations operates in time bounded by a polynomial function of the size of the input, but this polynomial has constant coefficients that grow exponentially fast as a function of the size of A.

Because idempotent algebras are rather special, some of the results we have proved for them do not extend directly to all finite algebras. For any finite algebra $\mathbf{A}$, it is true of course that $\mathbf{A}$ has a cube-term iff the idempotent reduct of $\mathbf{A}$ has no cube-term blocker. This is not a satisfactory characterization, because the relation between operations $f_1, \ldots, f_n$ over A, and the clone of all idempotent operations belonging to the clone generated by $f_1, \ldots, f_n$ is not at all well understood. One may get a glimpse of the possible difficulties that must be overcome to understand this relation, by carefully studying the following example.

We proved that if $\mathbf{A}$ and $\mathbf{B}$ are finite idempotent algebras of the same signature, and each has a cube-term, then $\mathbf{A} \times \mathbf{B}$ has a cube-term. (See the proof of Corollary 2.5.) This is not true for non-idempotent algebras. S. V. Polin's example of a finitely generated variety that satisfies some non-trivial congruence identity, but is not congruence modular, is the variety generated by the algebra $\mathbf{A} \times \mathbf{B}$ where $\mathbf{A} = \langle \{0, 1\}, \wedge, f_1, g_1 \rangle$ and $\mathbf{B} = \langle \{0, 1\}, \wedge, f_2, g_2 \rangle$, $f_1(x) = x$, $f_2(x) = 1 - x$, $g_1(x) = 1 - x$, $g_2(x) = 1$ and $x \wedge y$ equal to the smaller of x and y in both algebras. Each of the algebras $\mathbf{A}$ and $\mathbf{B}$ is term-equivalent to the two-element Boolean algebra, so they each have a cube-term and finitely related clone. The algebra $\mathbf{A} \times \mathbf{B}$ does not have Day terms (does not generate a congruence modular variety), and so it cannot have a cube-term. Incidentally, one can show that $\mathbf{A} \times \mathbf{B}$ is finitely related.

4. Finitely related idempotent algebras in congruence modular varieties

As we mentioned in the introduction, M. Valeriote has conjectured that every finitely related finite algebra that has Day terms must have a cube-term. (Conversely, we know that a finite algebra with a cube-term is finitely related and has Day terms.) This conjecture is true iff it is true for idempotent algebras, since cube-terms and Day-terms are idempotent operations. Thus in considering this conjecture, we shall continue to keep our focus almost exclusively on idempotent algebras.

Here is a special characterization of the finite idempotent algebras in congruence modular varieties that do have cube-terms, which could turn out to be of use in settling Valeriote's conjecture. It has the virtue of introducing some interesting problems that possibly no-one has had reason to consider before.

Theorem 4.1. *Let $\mathbf{A}$ be a finite idempotent algebra in a congruence modular variety. $\mathbf{A}$ has a cube-term if and only if for every subalgebra $\mathbf{E}$ of $\mathbf{A}^2$, every polynomial expansion of a homomorphic image of $\mathbf{E}$ is finitely related.*

Proof. Let $\mathbf{A}$ be a finite idempotent algebra in a congruence modular variety. Since the class of finite idempotent algebras with cube-terms is closed under taking subalgebras, homomorphic images, finite powers and expansions, our task reduces to showing that if $\mathbf{A}$ has a cube-term blocker then we can find a not finitely related algebra as described. Let S be minimal (under inclusion) among all the subuniverses of $\mathbf{A}$ that have no cube-term. Then by Theorem 2.1, S has a subuniverse D, $\emptyset \neq D < S$, such that (D, S) is a cube-term blocker.

Let θ be a maximal proper congruence of $\mathbf{S}$. Since $\mathbf{S}$ is idempotent, and whenever $d \in D$ and $s \in S \setminus D$ then $\{d, s\}$ generates S, it follows that D is a union of θ-classes. Thus $\mathbf{S}' = \mathbf{S}/\theta$ has a cube-term blocker (D', S'), $D' = D/\theta$. Consequently, the algebra $\mathbf{S}'$ is a simple, non-Abelian algebra. (Every Abelian algebra in a congruence modular variety has a Maltsev term, which is a cube-term.) From commutator theory, it follows that $\mathbf{S}'$ is neutral, i.e., every finite subdirect power of $\mathbf{S}'$ has a distributive congruence lattice. (Confer [7] Chapter 8, Exercise 2, pages 89 and 199.)

Let $\mathbf{R}$ be the subalgebra of $\mathbf{S}' \times \mathbf{S}'$ with universe $\{(x, y) \in S' \times S' : \{x, y\} \cap D' \neq \emptyset\}$. Define $\mathbf{T}$ as the polynomial expansion of $\mathbf{R}$ obtained by adding all the constants as operations. Now $\mathbf{T}$ is neutral, and due to the presence of the constants, the free algebra on three generators in the variety generated by $\mathbf{T}$ is a finite subdirect power of $\mathbf{T}$. Thus $\mathbf{T}$ belongs to a congruence distributive variety.

Now $\mathbf{T}$ is a polynomial expansion of a homomorphic image of a subalgebra of $\mathbf{A}^2$. Let us assume that $\mathbf{T}$ is finitely related. Then by a result of L. Barto [2], $\mathbf{T}$ has a near-unanimity operation $N(x_1, \dots, x_k)$ among its term operations, for some $k \geq 3$. We can express $N(\bar{x})$ as a polynomial operation of $\mathbf{R}$, say

$$N(x_1, \dots, x_k) = t(a_1, \dots, a_m; x_1, \dots, x_k)$$

where $t(\bar{y}; \bar{x})$ is a term operation and $a_1, \dots, a_m$ are elements, of the algebra $\mathbf{R}$. Choose $d \in D'$ and $s \in S' \setminus D'$, and put $e_1 = (d, s)$, $e_2 = (s, d)$.

There is a special variable of the term $t(\bar{y}; \bar{x})$ with the property that when all the variables are evaluated in S' with that variable taking a value in D', then t gives a value in D'. Since N gives the value e_2 evaluated at any tuple where all but one of x_i are assigned e_2 and the remaining variable is assigned e_1, it follows that the special variable cannot be among $x_1, \dots, x_k$. So it is say, y_j. Let $a_j = (p, q)$. Since $(p, q) \in R$ we may, without loss of generality, assume that $p \in D'$. But then it follows that the range of $N(\bar{x})$ is entirely contained in $D' \times S'$, which contradicts the fact that N acting in T does take e_2 as a value. This establishes that $\mathbf{T}$ is not finitely related, and finishes our proof of the theorem. •

We thank the referee for the simplification of the proof of the next lemma and its corollary.

Lemma 4.2. *Let* $\mathbf{R}$ *and* $\mathbf{C}$ *be finite algebras such that* $\mathbf{R} \leq \mathbf{C}^n$ *and for some* i *the* i*-th projection of* $\mathbf{R}$ *equals* $\mathbf{C}$*. Then* $\mathbf{C}$ *is finitely related iff* $\mathbf{R}$ *is finitely related.*

Proof. Suppose that $\mathbf{R} \leq \mathbf{C}^n$, and without loss of generality that the first projection of $\mathbf{R}$ equals $\mathbf{C}$, where $\mathbf{C}$ is a finite, finitely related algebra. There is, for some k, a k-ary relation ρ such that the clone of all term operations of $\mathbf{C}$, or $\operatorname{Clo}\mathbf{C}$, is identical with the set of all operations over C that respect ρ.

We define a finite collection of admissible relations for $\mathbf{R}$. For $1 \leq i \leq n$, we define $\pi_i : \mathbf{R} \to \mathbf{C}$ to be the projection at the i-th coordinate. Then we put

$$\rho_i = \{(x_1, \dots, x_k) \in R^k : (\pi_i(x_1), \dots, \pi_i(x_k)) \in \rho\}\,.$$

For $1 \leq i, j \leq n$ we define $\varepsilon_{ij} = \{(x, y) \in R^2 : \pi_i(x) = \pi_j(y)\}$. We put

$$\Sigma = \{\rho_i : 1 \leq i \leq n\} \cup \{\varepsilon_{ij} : 1 \leq i, j \leq n\}\,.$$

Clearly the clone $\operatorname{Clo}\mathbf{R}$ is contained in the clone $\Sigma^\perp$ of all operations that respect all the relations in Σ—i.e., the relations of Σ are admissible relations for $\mathbf{R}$.

On the other hand, if F is an m-ary operation in $\Sigma^\perp$, then F respects the projection congruences ε_{ii}, and thus we can write $F = (F_1 \times \cdots \times F_n)|_R$ where each F_i is an m-ary operation over $\pi_i(R)$. Since F respects ε_{1i}, and π_1 is surjective, then $F_i = F_1|_{\pi_i(R)}$ for all i. Thus $F = (G \times \cdots \times G)|_R$, $G = F_1$. Finally, since F respects ρ_1 and π_1 is surjective, it follows that G respects ρ. This means that $G \in \operatorname{Clo}\mathbf{C}$, and hence $F \in \operatorname{Clo}\mathbf{R}$.

We have proved that $\operatorname{Clo}\mathbf{R} = \Sigma^\perp$; consequently, $\mathbf{R}$ is finitely related.

To prove the other direction suppose that $\mathbf{R}$ is finitely related via a relation α of $\mathbf{R}$. Take an arbitrary admissible relation ρ of $\mathbf{C}$. As before, ρ_1 is an admissible relation of $\mathbf{R}$, thus it is in the relational clone generated by α, and definable by a primitive positive formula. The relations ρ_1 and α can be viewed as admissible relations of $\mathbf{C}$, hence ρ_1 is in the relational clone generated by α over $\mathbf{C}$, as well. But ρ is obtained from ρ_1 by projecting it to some coordinates. Hence ρ is in the relational clone generated by α over $\mathbf{C}$. Thus $\mathbf{C}$ is finitely related via the relation α. •

The following corollary has been independently proved by B. A. Davey, M. Jackson, J. G. Pitkethly and Cs. Szabó [6].

Corollary 4.3. *If $\mathbf{A}$ is a finite algebra, $n \geq |A|$ and $\mathbf{F}$ is the free algebra in the variety generated by $\mathbf{A}$ with an n-element free generating set, then $\mathbf{A}$ is finitely related iff $\mathbf{F}$ is finitely related. Consequently, if $\mathbf{A}$ and $\mathbf{B}$ are similar finite algebras such that $V(\mathbf{A}) = V(\mathbf{B})$, then $\mathbf{A}$ is finitely related iff $\mathbf{B}$ is finitely related.*

Proof. The n-generated free algebra in the variety generated by $\mathbf{A}$ can be represented by the set of n-ary term operations of $\mathbf{A}$ as a subpower $\mathbf{F} \leq \mathbf{A}^{|A^n|}$ generated by the projections. If $n \geq |A|$ and $A = \{a_1, \dots, a_n\}$, then the projection of $\mathbf{F}$ to its $(a_1, \dots, a_n)$ coordinate is full, therefore the previous lemma can be applied. •

Lemma 4.4. *Suppose that $\mathbf{A}$ is an algebra such that every expansion of $\mathbf{A}$ (by adding possibly infinite many new operations) is finitely related. Then every subalgebra and every homomorphic image of $\mathbf{A}$ also has the property that all expansions are finitely related.*

Proof. It suffices to show that subalgebras and quotients of $\mathbf{A}$ are finitely related.

First, let $\mathbf{S}$ be a subalgebra of $\mathbf{A}$. To see that $\mathbf{S}$ is finitely related, we define a clone on A. Let $\mathcal{C}$ be the clone of all operations F over A such that F respects S (i.e., S is a subuniverse of the algebra $\langle A, F\rangle$), and there is some $G \in \operatorname{Clo}\mathbf{A}$ so that $F|_S = G|_S$. Clearly, $\langle A, \mathcal{C}\rangle$ is an expansion of $\mathbf{A}$, so is finitely related. This means that there is a finite set Γ of relations over A such that $\mathcal{C} = \Gamma^\perp$.

First, notice that for any retraction $e : A \to S$, i.e., for any function $e : A \to A$ such that $e(A) = S$ and $e(x) = x$ for all $x \in S$, and for any $\rho \in \Gamma$, we have that $e(\rho) \subseteq \rho$—i.e., $(x_1, \dots, x_k) \in \rho$ implies $(e(x_1), \dots, e(x_k)) \in \rho$. This is true because $e \in \mathcal{C}$. Next, notice that for $\rho \in \Gamma$, and any retraction e as above, we have $e(\rho) = \rho|_S$; thus $e(\rho)$ is independent of the choice of e.

Now choose a retraction e of A onto S, and put

$$\Sigma = \{e(\rho) : \rho \in \Gamma\}.$$

We claim that the clone of $\mathbf{S}$, $\operatorname{Clo}\mathbf{S}$, is identical with $\Sigma^\perp$. That $\operatorname{Clo}\mathbf{S} \subseteq \Sigma^\perp$ follows from the fact that $\operatorname{Clo}\mathbf{A} \subseteq \Gamma^\perp$. Conversely, let $F : S^k \to S$ be any member of $\Sigma^\perp$. Define $G : A^k \to A$ by $G(x_1, \ldots, x_k) = F(e(x_1), \ldots, e(x_k))$. If $\rho \in \Gamma$, then because ρ is admissible for e and $e(\rho)$ is admissible for F, it follows that ρ is admissible for G. Thus $G \in \Sigma^\perp = \mathcal{C}$. This implies that $F = G|_S = H_S$ for some $H \in \operatorname{Clo}\mathbf{A}$.

We have now proved that $\Sigma^\perp \subseteq \operatorname{Clo}\mathbf{S}$. It follows that $\Sigma^\perp = \operatorname{Clo}\mathbf{S}$. This establishes that $\mathbf{S} \in \mathcal{F}$.

Now let θ be any congruence relation of $\mathbf{A}$. We must show that $\mathbf{B} = \mathbf{A}/\theta$ is finitely related. We proceed as before, by first defining an appropriate expansion of $\mathbf{A}$. Namely, we now define $\mathcal{C}$ to be the clone of all operations F such that F respects θ and $F/\theta = G/\theta$ for some $G \in \operatorname{Clo}\mathbf{A}$. Here by F/θ we mean the operation defined by

$$(F/\theta)(x_1/\theta, \ldots, x_n/\theta) = F(x_1, \ldots, x_n)/\theta.$$

As before, $\langle A, \mathcal{C}\rangle$ is an expansion of $\mathbf{A}$; and so there is a finite set Γ of relations over A such that $\mathcal{C} = \Gamma^\perp$.

Letting $\pi : \mathbf{A} \to \mathbf{A}/\theta$ be the quotient homomorphism, we put

$$\Sigma = \{\pi(\rho) : \rho \in \Gamma\}.$$

We claim, naturally, that $\operatorname{Clo}(\mathbf{A}/\theta) = \Sigma^\perp$.

Notice that if $s : A/\theta \to A$ is any choice function, i.e., a function such that $s(P) \in P$ for each θ-equivalence class P, then $s \circ \pi \in \mathcal{C}$ and so $s\pi(\rho) \subseteq \rho$ for all $\rho \in \Gamma$. Choose such a function s.

Now an obvious argument shows that every term operation of $\mathbf{A}/\theta$ respects the relations in Σ. Conversely, suppose that $F : (A/\theta)^k \to A/\theta$ respects the relations in Σ. Define $G : A^k \to A$ by

$$G(x_1, \ldots, x_k) = s(F(x_1/\theta, \ldots, x_k/\theta)).$$

We need to show that $G \in \mathcal{C}$. So let $\rho \subseteq A^m$ be any member of Γ. Suppose that $\bar{x}^1, \ldots, \bar{x}^k \in \rho$. Letting $g_i = G(x_i^1, \ldots, x_i^k)$ for $1 \le i \le m$, we need to show that $\bar{g} = (g_1, \ldots, g_m) \in \rho$. Since $\pi(\bar{x}^j) \in \pi(\rho)$ and F respects $\pi(\rho)$, it follows that where $h_i = F(x_i^1/\theta, \ldots, x_i^k/\theta)$ and $\bar{h} = (h_1, \ldots, h_m)$, we have $\bar{h} \in \pi(\rho)$. So there is $(x_1, \ldots, x_m) \in \rho$ so that

$$(h_1, \ldots, h_m) = (\pi(x_1), \ldots, \pi(x_m)).$$

Note that $g_i = s(h_i)$, so that

$$\bar{g} = (g_1, \ldots, g_m) = (s\pi(x_1), \ldots, s\pi(x_m)) \in \rho,$$

as desired. Thus indeed, $G \in \mathcal{C}$.

Now it follows that there exists $H \in \operatorname{Clo}\mathbf{A}$ so that $G/\theta = H/\theta$. It remains to note that $G/\theta = F$. Indeed,

$$(G/\theta)(x_1/\theta, \ldots, x_k/\theta) = \pi(G(x_1, \ldots, x_k)) =$$
$$= \pi s(F(\pi(x_1), \ldots, \pi(x_k))) = F(x_1/\theta, \ldots, x_k/\theta).$$

This concludes our proof that $\mathbf{A}/\theta$ is a finitely related algebra. •

5. Concluding remarks and conjectures

We hope that the results uncovered in this paper will help point the way to a resolution of Valeriote's conjecture. It would be very nice if that is true. Among the consequences would be that the constraint satisfaction problem over a finite relational structure $\mathbf{R}$ is tractable if $\mathbf{R}$ has Day operations in its algebra of polymorphisms.

The Theorems 3.4 and 4.1 and Lemma 4.2 above yield as a corollary the next theorem.

Theorem 5.1. *Valeriote's conjecture is equivalent to each of the statements below.*

(1) *The class of finite and finitely related idempotent algebras generating congruence modular varieties is closed under idempotent expansions.*
(2) *The class of finite and finitely related idempotent algebras generating congruence modular varieties is closed under forming subalgebras and homomorphic images and every full polynomial expansion of such an algebra is finitely related.*

Our results raise many questions that we have not been able to answer. Letting $\mathbf{A}$ and $\mathbf{B}$ be finite algebras of the same signature, we ask for instance, which of these possible properties of $\mathbf{A}$ and $\mathbf{B}$ must be inherited by their direct product: finitely related, idempotent and finitely related, idempotent and possess Day terms? From Corollary 2.5, we know that $\mathbf{A} \times \mathbf{B}$ is finitely related and has Day terms if $\mathbf{A}$ and $\mathbf{B}$ are idempotent and each has a cube-term. Under what assumptions is finite relatedness inherited when passing to subalgebras and/or homomorphic images?

Suppose that $\mathbf{A}$ is a finite and finitely related algebra with Day terms such that the full expansion of $\mathbf{A}$ by constants has Jónsson terms. Must $\mathbf{A}$ have a cube-term among its polynomial operations?

Example of A. Krokhin. Here is an example, shown to us by A. Krokhin, of a finitely related idempotent algebra with a subalgebra that is not finitely related. It is $\mathbf{F} = \langle F, \mathcal{C} \rangle$ where $\mathcal{C}$ is the clone of all conservative operations on $F = \{0, 1, 2, 3\}$ that respect the relation

$$R = \{(x, y) \in F^2 : x \leq y \text{ and } (x, y) \neq (1, 3)\}.$$

An operation is conservative if it respects every unary relation over F. The subalgebra of $\mathbf{F}$ on the set $\{0, 1\}$ is not finitely related. In fact, it can be shown that the clone of this subalgebra is precisely $\mathcal{C}_{\mathcal{I}}(\{0\}, \{0, 1\})$.

After this paper was written, we were informed that the paper B. A. Davey, M. Jackson, J. G. Pitkethly and Cs. Szabó [6] has examples showing that the finite relatedness property in finite algebras fails, in general, to be preserved by any of the algebraic constructions of forming direct products, subalgebras, or homomorphic images.

References

[1] E. Aichinger, R. McKenzie, P. Mayr, *Finite algebras with cube-terms are finitely related,* (manuscript).
[2] L. Barto, *CD implies NU,* (manuscript).
[3] L. Barto, M. Kozik, *New conditions for Taylor varieties and CSP,* Logical Methods in Computer Science (to appear).

[4] L. Barto, M. Kozik, *New conditions for Taylor varieties and CSP.* Proceedings of 25th IEEE Symposium on Logic in Computer Science, LICS'10, 100–109, 2010.
[5] J. Berman, P. Idziak, P. Marković, R. McKenzie, M. Valeriote, R. Willard, *Varieties with few subalgebras of powers,* Transactions AMS **362** (2009), 1145–1173.
[6] B. A. Davey, M. Jackson, J. G. Pitkethly and Cs. Szabó, *Finite degree: algebras in general and semigroups in particular,* Semigroup Forum (to appear).
[7] R. Freese, R. McKenzie, Commutator Theory for Congruence Modular Varieties, London Mathematical Society Lecture Note Series No. 125, Cambridge University Press, 1987.
[8] J. Horowitz, PhD dissertation, McMaster University.
[9] P. Idziak, P. Marković, R. McKenzie, M. Valeriote, R. Willard) *Tractability and learnability arising from algebras with few subpowers,* Proceedings of the 22nd Annual IEEE Symposium on Logic in Computer Science, 221–230.
[10] P. Marković, R. McKenzie, *Few subpowers, congruence distributivity, and near-unanimity terms,* Algebra Universalis (to appear).
[11] M. Valeriote, *A subalgebra intersection property for congruence distributive varieties,* Canadian J. Math **61** (2009), no. 2, 451–464.

Department of Mathematics, University of Novi Sad, Serbia
E-mail address: `pera@im.ns.ac.yu`

Department of Mathematics, University of Szeged, Hungary
E-mail address: `mmaroti@math.u-szeged.hu`

Department of Mathematics, Vanderbilt University, Nashville, U.S.A.
E-mail address: `rn.mckenzie@vanderbilt.edu`